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Statistical Tolerance Analysis Using GapSpace

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Abstract: This paper describes a new method for statistical tolerance analysis using the GapSpace model. We show how, after modeling the distribution of manufactured dimensions with a joint probability density function, assembly analysis is performed by integrating the density function over a tolerance region or assembly region described in GapSpace. Through an example, this paper addresses how statistical tolerance analysis should mimic the "manufacturing scheme" of the parts and how the GapSpace model can be used to identify optimal results.
Keywords: statistical tolerancing, assembly modeling, assembly analysis, GapSpace

1. INTRODUCTION

Tolerance analysis is a continued focus of the manufacturing industry as it seeks to both improve the quality of its products and produce them with lower cost. The appropriate allocation of tolerances in an assembly may result in lower costs per assembly and a higher probability of good fit, reducing the number of rejects or the amount of rework. Statistical tolerance analysis is a powerful analytical method because it not only predicts the effect of manufacturing variation on design performance and production cost, but also allows designers and manufacturing personal to take advantage of statistical averaging to relax the component tolerances without sacrificing quality.

The “GapSpace Model” proposed by Morse [Morse 99] can be used to capture necessary and sufficient conditions for the satisfaction of various assembly criteria. The problem of interchangeable assembly is represented by partitioning the GapSpace into regions where assembly is possible and regions where it is not. The model of an individual assembly or a population of potential assemblies can be located in the GapSpace and be analyzed by comparison to these regions.

In this paper we distinguish between several types of parameter assignments, or dimension schemes. The design scheme describes how dimensions and tolerances are specified for the parts, the manufacturing scheme captures the dimensions that vary due to the manufacturing process, and the inspection scheme identifies the dimensions that are measured on the finished components. The manufacturing scheme will provide the best information for statistical assembly analysis, although it may not be available during the initial design phase. Different manufacturing schemes with the same design scheme may generate huge differences in statistical tolerance analyses. In this paper, this difference is quantitatively and qualitatively shown using the GapSpace model.

In the next section we identify a few selected works in assembly analysis, and relate these works to our research. The GapSpace model is described briefly using an example, as it is the basis for later analysis. We then present the execution of statistical tolerance analysis in GapSpace and show how the optimization results are realized.

2. RELATED WORK

The linear "stack-up" method is a fundamental tolerance analysis technique, as described in [Bjørke 89] and others. This analysis requires that assembly components be in contact at mating faces and closed loops are generated through these faces. Every loop is composed by chain of dimensions and a single gap. The "worst case" tolerance analysis is performed by accumulation of the variability in the chain. In 1-D analysis, all sensitivities of dimensions are ± 1. The RSS (Root Sum Squared) model and its complex forms [Bender 62] are widely used for statistical tolerance analysis. We adopt the "RSS" model in the paper, in which the distributions of part dimensions are assumed to be normal.

The group led by K.W. Chase at Brigham Young University has worked to extend this stack-up analysis to 2- and 3-dimensional assemblies [Chase 97][Gao 95]. They use vector loop-based assembly models for these analyses, in which closed vector loops describe the small kinematic adjustments and open vector loops describe critical clearances or other assembly features. A commercial software package based on this work is available and can perform worst-case and statistical analyses.

Monte Carlo simulation is commonly used for statistical tolerance analysis [Grossman 76]. A random number generator is used to simulate variability of each component's size and form. These values are combined through the assembly function to determine the resulting influence on some clearance or gap dimension. This simulation method can be quite time-consuming.

Mullins and Anderson [Mullins 98] have proposed a graph-based approach to tolerance analysis. By identifying constraints that exist among components of an assembly, the Physically Constraining Face Set (PCFS) is generated. Loops are constructed by working on the assembly graph and PCFS. In the GapSpace model used in this paper, the step for identifying PCFS is not necessary as these constrains are implicitly included in the rules of searching loops in the assembly graph.

With the exception of the PCFS model, the methods described above require that there is contact between assembly components and at most one clearance exists in every loop. The GapSpace model does not have these restrictions. In the examples that follow, we show how assemblies without required contact conditions can be analyzed for both worst case and statistical criteria.

3. GAPSPACE MODEL

The GapSpace model allows us to capture the physical requirements in one-dimensional tolerancing analysis independently of how the parts are dimensioned. It can be used to induce whether particular dimension schemes impose sufficient conditions for assembly. This section introduces GapSpace model, as it is the basis of other sections of the paper.

Two basic concepts, *directed dimension tree* and *liaison*, are used throughout this paper and need to be explained. The directed dimension tree is a data structure that represents the dimension scheme for parts in a single direction. A directed dimension tree describes a unique set of feature relationships, although a single part may have multiple equivalent dimension trees, as shown in Figure 1.

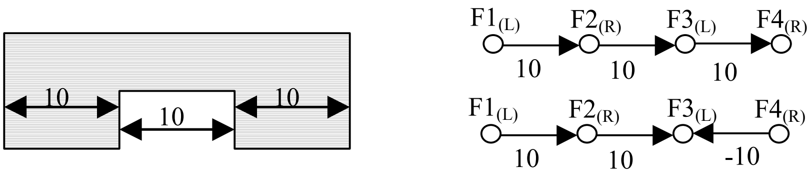


Figure 1; A nominal part and two equivalent directed dimension trees.

A liaison is defined as the specification of adjacency for a pair of opposing features, which are from different parts and may potentially interfere when assembling. The value of a liaison is the signed distance between the two features. The hinge example in Figure 2 is used in this paper to show how the GapSpace model works. There are four liaisons in Figure 2, which are named g_1, g_2, g_3 and g_4. Treating the gaps as independent variables, we can construct a four-dimensional linear vector space with basis vectors $[1\ 0\ 0\ 0]^T$, $[0\ 1\ 0\ 0]^T$, $[0\ 0\ 1\ 0]^T$, and $[0\ 0\ 0\ 1]^T$. This linear vector space is called the GapSpace for the assembly.

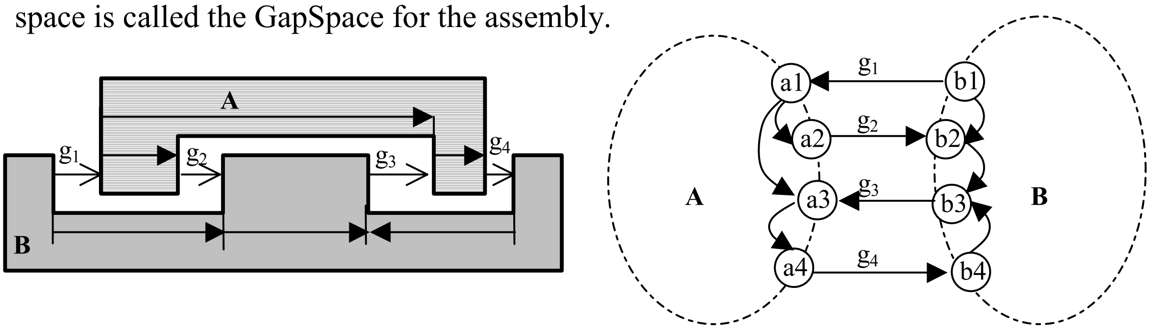


Figure 2; A hinge assembly and its assembly graph.

After identifying all liaisons and dimension trees, an assembly graph can be constructed. In the assembly graph, liaisons are represented with directed arcs between the parts which, in turn, are represented using dimension trees. Assembly graphs can have different levels of detail. For instance, the low-level graph in Figure 3, called a liaison graph, has each part regarded as a node and the arcs in the graph are the liaisons between parts. Higher-level graphs explicitly show the directed dimension trees for each part. Certain analyses can be thought of as "dimensioning independent" and only require the information captured in the liaison graph.

Based on principles detailed in [Morse 99], assembly cycles are determined by following the directed arcs in the liaison graph. *Fits Conditions* (FCs) correspond to these assembly cycles and represent the physical requirements for assembly. Four assembly cycles exist in the hinge example, so four FCs are generated. These FCs must be non-negative to satisfy the assembly requirement; this is shown in equations (1).

$$\begin{cases} FC1 = g1 + g2 \geq 0 \\ FC2 = g2 + g3 \geq 0 \\ FC3 = g3 + g4 \geq 0 \\ FC4 = g1 + g4 \geq 0 \end{cases} \quad (1)$$

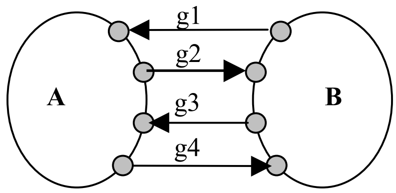


Figure 3; Liaison graph of Figure 2.

The four fits conditions describe a subspace of the GapSpace spanned by the vectors $[1\ \ 1\ \ 0\ \ 0]^T$, $[0\ \ 1\ \ 1\ \ 0]^T$, $[0\ \ 0\ \ 1\ \ 1]^T$, and $[1\ \ 0\ \ 0\ \ 1]^T$ corresponding to the fits condition gap pairs. This subspace is named the assembly space. The assembly space for the hinge example is shown in Figure 4a. The assembly region corresponds to the portion of this subspace in which the inequalities from equation (1) are satisfied; it is independent of part representation and captures the necessary and sufficient conditions required for assembly.

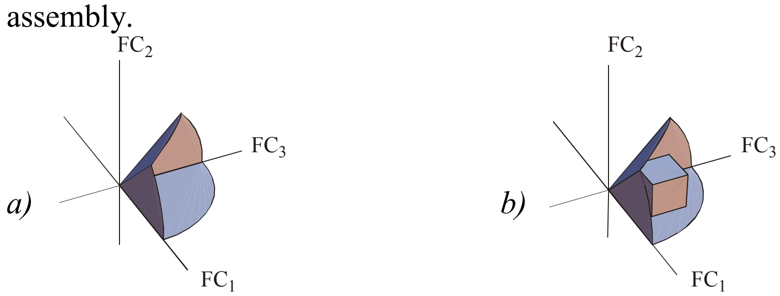


Figure 4; Assembly region of a hinge and a tolerance region guaranteeing assembly.

Specific dimension configurations and tolerances applied to the assembly result in a region in the assembly space representing the possible dimension combinations permitted by the design specification. This region is called the tolerance region of the assembly. Figure 4b shows a tolerance region contained within the assembly region. The tolerance region for a single part can be generated by applying tolerances to that part while maintaining fixed nominal dimensions for all other parts in the assembly.

4. STATISTICAL TOLERANCE ANALYSIS

Statistical tolerance analysis can be used by designers and manufacturing personnel to take advantage of statistical averaging over assemblies of parts, allowing the use of less restrictive tolerances in exchange for admitting the small probability of non-assembly. In this section we show how statistical analysis is facilitated by the GapSpace model.

We discuss first the manufacturing dimension scheme and its representation of the statistical variability of dimensions. Then we describe the role of the tolerance regions and assembly regions in statistical analysis. We examine statistical independence of the dimensions and the errors incurred by an incorrect assumption of independence. An optimization result is obtained by assigning dimensions effectively. Finally, we compare different manufacturing schemes to determine which predicts the highest assembly yield.

4.1. Manufacturing dimension scheme

The manufacturing dimension scheme of a single part is a parametric representation of the variability imposed by the manufacturing process. In the scheme, certain dimensions vary as a function of process parameters while other dimensions are dependent on these 'primary' dimensions.

Because of manufacturing process limitations, the manufacturing dimension scheme may not be fully consistent with the design scheme. For example – the geometric position tolerance applied at MMC in Figure 5 allows additional position variability as the feature sizes depart from MMC. However, many processes that could be used to manufacture the features have independent factors which affect the size and the position of the features. It is difficult to envision a realistic manufacturing process that will automatically admit more positional variation as the features depart from MMC.

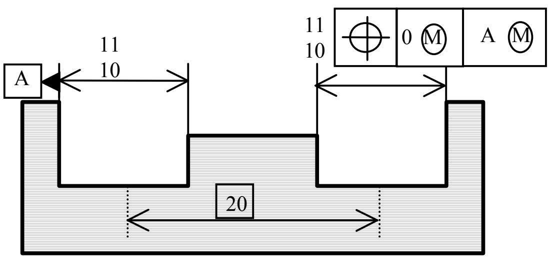
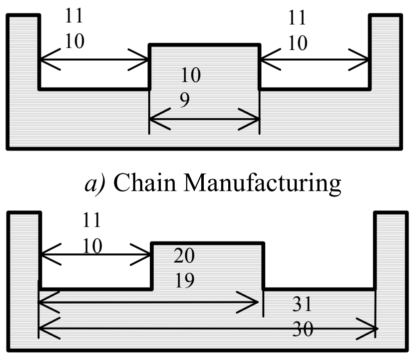


Figure 5; GD&T design scheme.



a) Chain Manufacturing

b) Baseline Manufacturing

Figure 6; Manufacturing schemes.

To manufacture the part specified in Figure 5, two types of manufacturing processes might be chosen, reflected by the manufacturing schemes shown in Figure 6a and 6b, which are the chain and baseline manufacturing processes respectively. Every dimension labeled in Figure 6 is assumed to be the output of one independent process. The range of the dimensions reflects an initial setup by the manufacturing engineer corresponding (he hopes) to ±3σ of the process for creating that dimension.

4.2. Representation

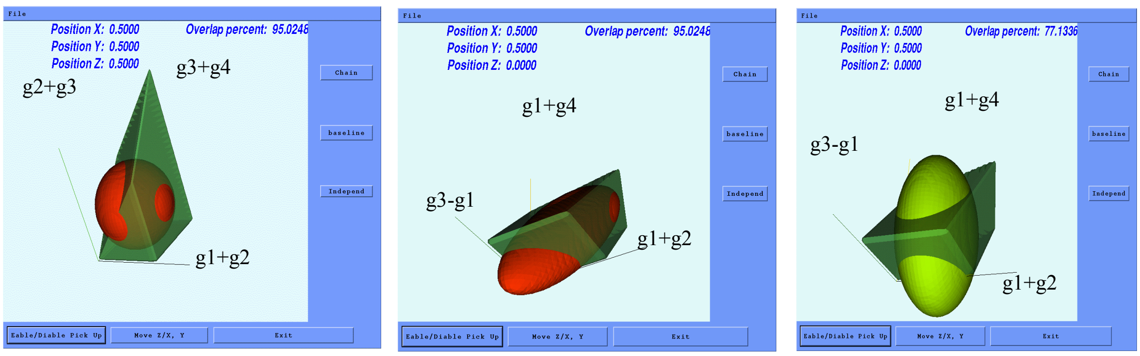
After assuming process parameters for each manufactured dimension, we can construct a joint probability density function (joint pdf) that represents the probability of every combination of dimension values. Once the joint pdf has been constructed, the probability of a part meeting its tolerance specification or an assembly satisfying a set of assembly criteria can be calculated by integrating the pdf over its tolerance region or assembly region respectively.

The probability that an individual part **x** satisfies its tolerance specification and the probability that an assembly **y** satisfies its assembly criteria are shown in equations (2) and (3) respectively. In these equations, TR represents the tolerance region, AR represents the assembly region, the B_i are the basis vectors for the assembly space, and the number of integration symbols is the dimension of the assembly space. The joint pdfs p_x and p_y depend on the distributions of the component dimensions in the assembly.

$$P(\text{x meets tolerance spec}) = \int \cdots \iint_{TR} p_x(B1, B2, \cdots) d\text{TR} \qquad (2)$$

$$P(\text{y meets assembly criteria}) = \int \cdots \iint_{AR} p_y(B1, B2, \cdots) d\text{A}R \qquad (3)$$

If the three dimensions for chain manufacturing process in Figure 6a exhibit independent normal statistics with the same variance, the joint pdf can be represented with spherical iso-probability surface as shown in Figure 7a. The assembly for this example is comprised of the part specified in Figure 5 and the nominal complement part from Figure 1. Because the basis vectors for the assembly space are each influenced by only one of the chain dimensions, the chain dimensions are orthogonal in this representation, hence the spherical surface. The tolerance region for the assembly is the polyhedron shown on Figure 7a. The fractional containment of the joint pdf within the tolerance region represents the probability that the part with the manufacturing process will satisfy the tolerance specifications.



a) *b)* *c)*

Figure 7; a) Joint pdf for chain manufacturing process contained in tolerance region.
b) Joint pdf for baseline manufacturing process contained in tolerance region.
c) Joint pdf when misunderstanding the independence of the dimensions.

4.3. Assessment

In order to determine if the manufacturing process follows the manufacturing dimension scheme, we would perform measurements on a subset of population to estimate the mean and variance of the population dimensions according to an inspection scheme. If we assume the inspection scheme is the same as the manufacturing scheme shown in Figure 6a, a sufficient random sample of parts will allow us to estimate that the dimensions have approximately the same variance, and correlation between them is near zero. The estimated probability of conformance to the tolerance specification will be the same as we determined from the analysis of the manufacturing scheme.

But what if the inspection process is performed per the baseline inspection scheme in Figure 6b, but the parts are manufactured as the chain scheme in Figure 6a? If we estimate the covariance terms between the baseline dimensions in addition to the means and variances of the individual dimensions, we will find these terms are no longer near zero. The joint pdf will not be spherical when we plot it in a coordinate system where the baseline dimensions are orthogonal. However, the proportion of the pdf containing in the tolerance region will keep the same because the tolerance zone is distorted also as shown in Figure 7b.

The distortion of the tolerance region and the joint pdf comes from the linear transformation from chain dimensions to baseline dimensions. The matrix C in equation (4) is used to implement the transformation in the hinge example, where $\mathbf{csy_{chain}}$ and $\mathbf{csy_{Baseline}}$ are the basis vectors for the different coordinate systems.

$$\begin{pmatrix} g1 + g2 \\ g1 + g4 \\ g3 - g1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} g1 + g2 \\ g3 + g4 \\ g2 + g3 \end{pmatrix}, \text{ or } \quad csy_{Baseline} = C \bullet csy_{chain} \tag{4}$$

If the distribution of the chain dimensions is a multivariate normal distribution, we can apply the C matrix as shown in Equation (5) to transform the distribution into the baseline coordinate system.

$$\text{If pdf}_{chain} \sim \text{N}[\mu, \Sigma] \text{ and pdf}_{Baseline} = C \bullet pdf_{chain}, \text{ then pdf}_{Baseline} \sim \text{N}\left[C\mu, C\Sigma C^T\right] \tag{5}$$

If, during inspection, we ignore the correlation between dimensions of the baseline inspection scheme, then we incorrectly estimate $\text{pdf}_{Baseline}$ ~N [Cμ, diag(CΣ CT)]. This distribution corresponds to treating each of the baseline dimensions manufactured independently as shown in Figure 7c. Note that far less of the pdf is contained in the tolerance region, and we would conclude (incorrectly) that fewer parts assemble.

4.4. Example

To quantify the statements about the containment of the distribution in the tolerance region, we provide the numerical values used to generate these figures. Let's use the same example in Figure 5 for GD&T design scheme, which will assemble with the nominal part in Figure 1. The manufacturing scheme of the part is the chain scheme in Figure 6a.

The joint distribution is shown on equation (6) and the iso-probability surface shown in Figure 7.a is drawn at the corresponding to the pdf value 7.05×10^{-6}. The variance 0.028 equals to $(range/6)^2$, where range is 1 from Figure 6a.

$$pdf_{chain} = \begin{pmatrix} g1+g2 \\ g3+g4 \\ g2+g3 \end{pmatrix} = \begin{pmatrix} C1 \\ C2 \\ C3 \end{pmatrix} \sim N\left[\begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} .028 & 0 & 0 \\ 0 & .028 & 0 \\ 0 & 0 & .028 \end{pmatrix}\right] \quad (6)$$

To find the probability that the part will satisfy the specified tolerance, we simply integrate the equation (2) using pdf_{chain} from equation (6). The resulting value is 95.0%.

We now consider the baseline representation of the tolerance region and joint pdf as shown in Figure 7.b. The transformed distribution $pdf_{Baseline}$ is listed in equation (7). The integration of the pdf over the tolerance region in baseline coordinates is also 95.0%. As we expected, the integration result is independent of the coordinate system.

$$pdf_{Baseline} = \begin{pmatrix} g1+g2 \\ g1+g4 \\ g3-g1 \end{pmatrix} = \begin{pmatrix} B1 \\ B2 \\ B3 \end{pmatrix} \sim N\left[\begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix}, \begin{pmatrix} .028 & .028 & -.028 \\ .028 & .083 & -.056 \\ -.028 & -.028 & .056 \end{pmatrix}\right] \quad (7)$$

If the inspection process uses baseline schemes, and the dimensions are unwittingly assumed to be independent. The incorrect baseline distribution behaves as equation (8). The integration of $pdf'_{Baseline}$ over the baseline tolerance region is 77.1%. The value equals the containment indicated in Figure 7c, which is much less than 95.0%.

$$pdf'_{Baseline} = \begin{pmatrix} g1+g2 \\ g1+g4 \\ g3-g1 \end{pmatrix} = \begin{pmatrix} B1' \\ B2' \\ B3' \end{pmatrix} \sim N\left[\begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix}, \begin{pmatrix} .028 & 0 & 0 \\ 0 & .083 & 0 \\ 0 & 0 & .056 \end{pmatrix}\right] \quad (8)$$

The difference between the two integration values comes from the accumulation of different independent dimensions for statistical analysis, so setting up the independence of dimensions correctly is critical for analysis.

4.5. Optimization result

Designers will always attempt to have higher containment, as that means the probability of a failed assembly will be lower. There are two ways to increase the containment for the above example. One is to decrease the variances of the manufacturing process. However, a side effect of this method is that the cost for the manufacturing process will increase. Another way is to choose the mean, or target, dimensions effectively. In the example given above, the means are assumed to be at the center of the dimension range. Changing the means can improve the containment without increasing the variances.

Because the equation (2) and (3) are not differentiable with respect to the means, we designed a program to search the optimization result by testing the increasing trend of probability related to the means. The optimization result is 98.29% in the example when the means becomes 10.57, 9.57 and 10.57 rather than 10.5, 9.5, and 10.5

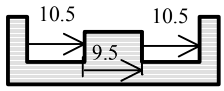
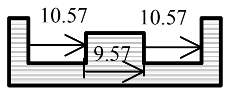
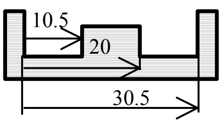
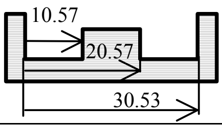
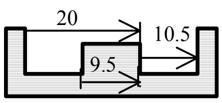
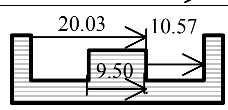
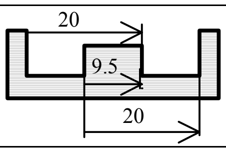
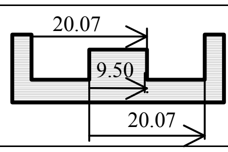
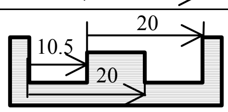
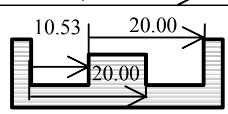
respectively. Only the mean vector of the joint distribution from equation (6) has been changed to obtain this result; the variances are unchanged.

4.6. Comparison by different manufacturing schemes

We discussed two manufacturing schemes as Figure 6 above. But the designer of manufacturing process may use different schemes to manufacture the part. In the table 1 below, we compare different schemes after assuming the standard deviation of each manufactured dimension is independent of the dimensional size and has the value 1/6 (using the normal distribution again). Dimensions labeled in the figures of Table I are the means, or "target" dimensions for each case.

The containment for each scheme is listed in Table I. The optimization result is calculated as described in section 4.5. The first scheme has the highest probability, which may be the best choice for the part designed in Figure 5. In high volume applications, a few percent increase in the yield can result in significant cost savings.

Table I; Assembly yield for different manufacturing schemes and optimization results.

Manufacturing scheme	**Probability in Tolerance region ($P_{(X\in TR)}$)**	**Optimization Result**	
		Dimensions changed	**Probability**
 	95.0%	 	98.3%
 	94.6%	 	96.2%
 	94.6%	 	96.1%
 	91.1%	 	92.8%
 	89.9%	 	91.8%

5. VTK IMPLEMENTATION

Using Visualization Toolkit (VTK) [Schroeder 97], we developed an application program for assembly analysis that generated the pictures in Figure 7. We can zoom, move and rotate the 3D objects in the pictures. We can also pick up and move the joint pdf object with respect to the tolerance region. The containment percentage and the center (mean) location are shown in the window and updated continuously with the movement of the joint pdf location. The optimization result above can also be found "manually" by moving the pdf object until finding the maximal containment.

6. SUMMARY

This paper begins by introducing GapSpace for one-dimensional analysis. We then show that the probability of satisfying assembly criteria or tolerance requirements can be calculated by integrating a joint pdf (representing the statistical variability of the parts) over the assembly region or tolerance region. Through a hinge example, we also show that the statistical analysis should correspond to the manufacturing scheme of the parts. Misunderstanding the independence of the dimensions in the manufacturing method may generate very large differences in the statistical analysis result. An optimization result shows that thoughtful dimension assignment can achieve higher assemblebility without having to improve the manufacturing process. An application using VTK is implemented to visualize the analysis process.

Future work is underway to extend the GapSpace model to 2-D and 3-D assembly analyses and to test different distributions (not only normal distribution) where appropriate to represent non-normal processes.

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