Topics

Substitute Geometry

Point Collection

Fitting to Points

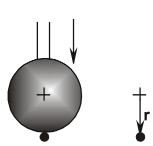
Fitting to Collections of Points

Exemplary Calculation

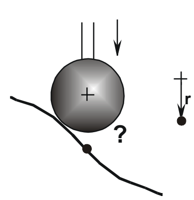
Comparison of Fitting Criteria

Extensions of Fitting Techniques

Measuring points: Compensate from stylus center.

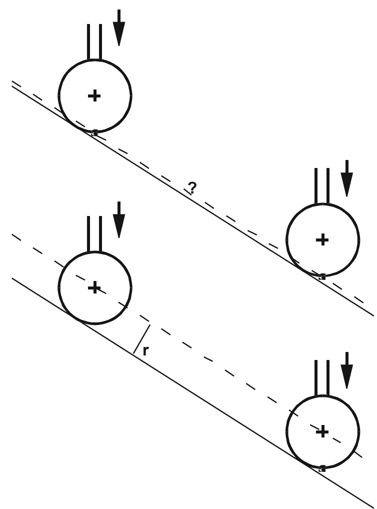


OK

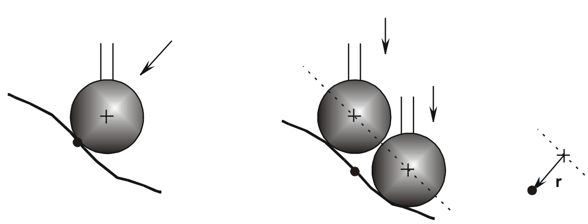


not OK

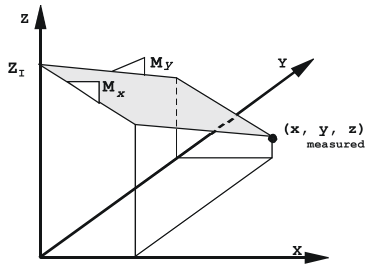
For certain features, we can compensate after fitting instead of before. These are the same features we talked about in the discussion of size: circles, planes, cylinders, etc.



If we know the surface normal, we can compensate in that direction. We can also sample an unknown surface to estimate the normal.



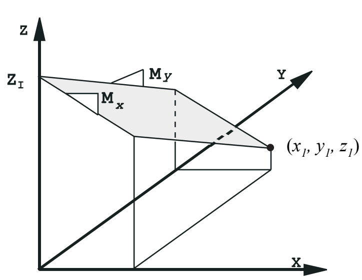
Fitting Features to Data: An Example



$$\begin{bmatrix} x & y & 1 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ Z_I \end{bmatrix} = \begin{bmatrix} z \\ \\ \end{bmatrix}$$

$$Z_I + x \cdot M_x + y \cdot M_y = z$$

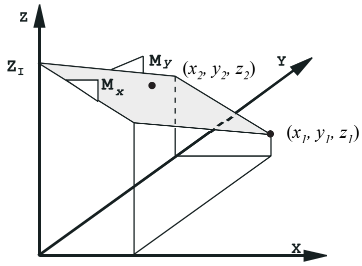
Fitting Features to Data: An Example



$$\begin{bmatrix} x_1 & y_1 & 1 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ Z_I \end{bmatrix} = \begin{bmatrix} z_1 \\ \\ \end{bmatrix}$$

$$Z_I + x \cdot M_x + y \cdot M_y = z$$

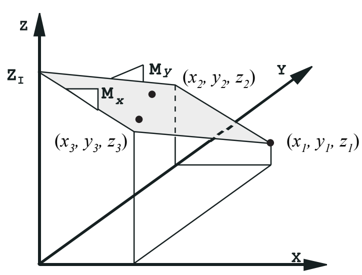
Fitting Features to Data: An Example



$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ & & \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ Z_I \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \ \end{bmatrix}$$

$$Z_I + x \cdot M_x + y \cdot M_y = z$$

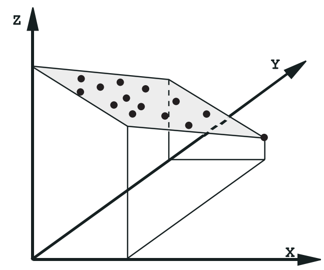
Fitting Features to Data: An Example



$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ Z_I \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$Z_I + x \cdot M_x + y \cdot M_y = z$$

Fitting Features to Data: An Example

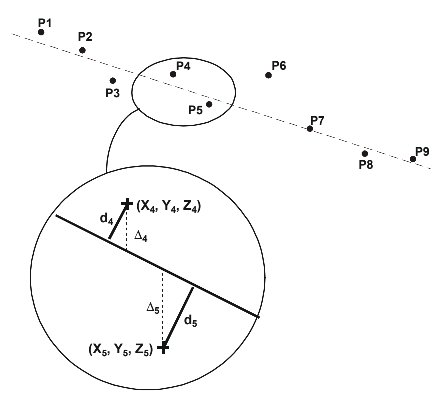


$$Z_I + x \cdot M_x + y \cdot M_y = z$$

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ Z_I \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ \vdots \\ z_n \end{bmatrix}$$

A x b

When there are more than the minimum number of points required to describe a feature, the fitted feature will have some error at each measurement point.



Instead of

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ Z_I \end{bmatrix} \neq \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ \vdots \\ z_n \end{bmatrix}$$

We have

$$\underbrace{\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} M_x \\ M_y \\ Z_I \end{bmatrix}}_{x} - \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ \vdots \\ z_n \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \vdots \\ \vdots \\ \Delta_n \end{bmatrix}}_{\Delta}$$

And the "best fit" plane will be the values of

$\{M_x, M_y, Z_I\}$ that give the **minimum** Δ values.

If the feature does not fit the points perfectly, we want to have the "best fit" in some way.

For a 3-point plane, we have $\begin{array}{c} Ax = b \\ or \\ Ax - b = 0 \end{array}$ and we can solve for *x*.

With more than 3 points, we have $Ax - b = \Delta$

where the Δ's are the z-error at each point.

What exactly to we want to minimize? We often minimize the "*p*-norm" of the Δ vector.

Consider the vector **x**: $\mathbf{x} = \{x_1, x_2, x_3, \cdots, x_n\}$

The p-norm of **x** is : $\|\mathbf{x}\|_p = \left(|x_1|^p + |x_2|^p + \ldots + |x_n|^p\right)^{\frac{1}{p}}$

Common values for p are (1, 2, and 8). What does this mean?

$$\|\mathbf{x}\|_1 = \left(|x_1|^1 + |x_2|^1 + |x_3|^1 + \cdots + |x_n|^1\right) = \sum |x_i|$$

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2} = \text{"}RSS\text{"}$$

$$\|\mathbf{x}\|_\infty = \left(|x_1|^\infty + |x_2|^\infty + |x_3|^\infty + \cdots + |x_n|^\infty\right)^{\frac{1}{\infty}} = \ ?$$

What's with this infinity norm?

$$\|\mathbf{x}\|_\infty = \left(|x_1|^\infty + |x_2|^\infty + |x_3|^\infty + \cdots + |x_n|^\infty\right)^{\frac{1}{\infty}} = \ ?$$

Let's say one of the x's has the largest absolute value. We'll call it x_*. Multiply the top and bottom of the equation by x_* to the 8 power.

$$\|\mathbf{x}\|_\infty = |x_*|^\infty \left(\left(\frac{|x_1|}{|x_*|}\right)^\infty + \left(\frac{|x_2|}{|x_*|}\right)^\infty + \left(\frac{|x_3|}{|x_*|}\right)^\infty + \cdots + \left(\frac{|x_*|}{|x_*|}\right)^\infty + \cdots + \left(\frac{|x_n|}{|x_*|}\right)^\infty \right)^{\frac{1}{\infty}}$$

Since all of the x's except x_* are less than x_*, each fraction goes to zero when raised to the 8 power. What we end up with is:

$$\|\mathbf{x}\|_\infty = |x_*|^\infty (0 + 0 + 0 + \cdots + 1 + \cdots + 0)^{\frac{1}{\infty}} = |x_*|^{\infty\left(\frac{1}{\infty}\right)} = |x_*|$$

The infinity norm simply reports the largest element in the vector.

If we want to minimize the errors using least squares, consider the following equations.

$$\|Ax - b\|_2 = \|\Delta\|_2$$

To minimize $\|\Delta\|_2$, *we must minimize* $\|Ax - b\|_2$

$$(Ax - b)^T (Ax - b) = (x^T A^T - b^T)(Ax - b)$$

$$= x^T A^T Ax - b^T Ax - x^T A^T b + b^T b$$

$$\tfrac{d}{dx}\left(x^T A^T Ax - b^T Ax - x^T A^T b + b^T b\right) = 2A^T Ax - 2A^T b$$

To find a minimum, we set the slope to zero, so

$$\tfrac{d}{dx}\left(x^T A^T Ax - b^T Ax - x^T A^T b + b^T b\right) = 2A^T Ax - 2A^T b = 0$$

Therefore

$$A^T Ax - A^T b = 0 \qquad \textit{or} \qquad A^T Ax = A^T b$$

Conclusion

To minimize $||Ax-b||_2$, solve $A^TAx = A^Tb$.

$$A^T A x = A^T b$$

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & \cdots & y_n \\ 1 & 1 & 1 & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ Z_I \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & \cdots & y_n \\ 1 & 1 & 1 & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ \vdots \\ z_n \end{bmatrix}$$

or

$$\begin{bmatrix} \sum x_i^2 & \sum x_i y_i & \sum x_i \\ \sum x_i y_i & \sum y_i^2 & \sum y_i \\ \sum x_i & \sum y_i & n \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ Z_I \end{bmatrix} = \begin{bmatrix} \sum x_i z_i \\ \sum y_i z_i \\ \sum z_i \end{bmatrix}$$

This is simple enough to do in a calculator, or excel.

Homework 10

- For the data posted on the web site, calculate the best fit planes, and report the largest + and – errors (Ax – b).