Lecture 2 – Motion of Particles: Rectilinear Cases

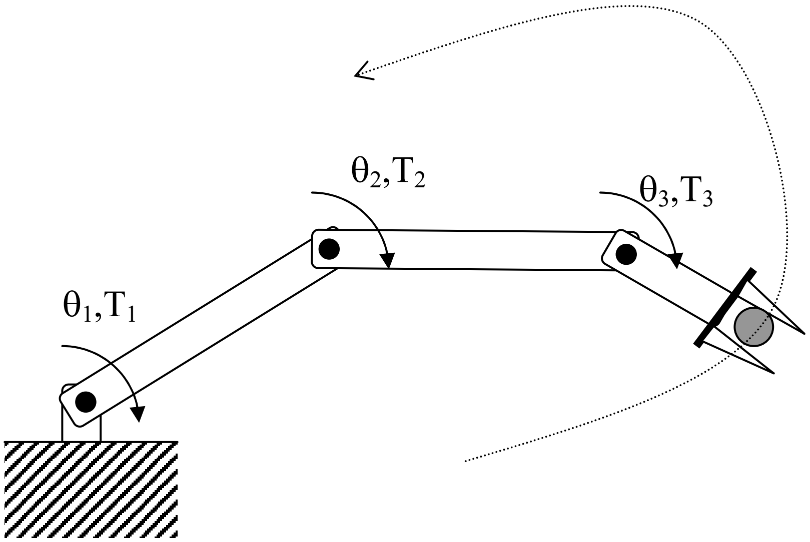
Administrative Issues

- Reading for Next Class: Go over Sample Problems 2/1, 2/2, 2/3, and 2/4
- Homework 1 Due Monday August 29 – Download from WEBCT

A. Introduction

Our study of dynamics will be divided into two categories. The first is the study of kinematics or the geometry of the motion of bodies. The second is the study of kinetics or the mechanism by which forces generate motion. The techniques of kinematics are particularly useful when some aspect of the motion of a body is known, measured or prescribed and we want to describe more about the characteristics of that motion. The techniques of kinetics are necessary when the motions of a particle or body are not known a priori, but the forces acting on a body are known and we want to predict the ensuing motions.

As an example, consider a robotic arm with three links/joints.



If the motors that move the joints are of sufficient power, they can generate nearly arbitrary angular positions at the joints without regard for the inertia of the links or the size of the payloads. In this case a study of the kinematics of the robot would allow us to calculate the relationship between the joint. angles, angular velocities and accelerations to the curvilinear motion of the end effector in space. If the motors are not of sufficient power, but instead provide some time dependent and limited torque, we may then have to calculate the dynamics of the robotic mechanism and hence the motion generated at the end effector. This motion will not be exactly what we want, but instead may deviate substantially from the desired path. The study of control systems deals with how to prescribe the proper torques necessary to generate an end effector path that is close to the desired path.

B. Kinematics of Particles

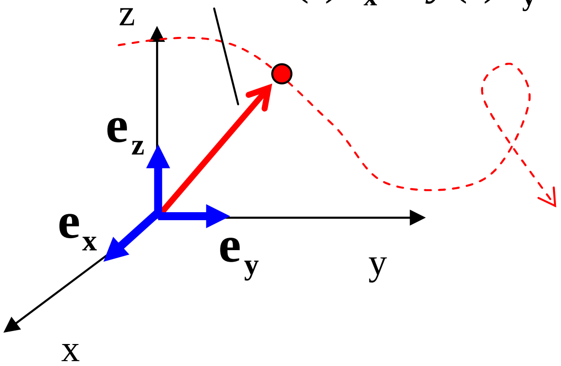
(Kinematics is a fancy word that refers to the "Geometric Aspects of Motion")

We will begin with the study of kinematics by looking at the simplified case of particle motion. A particle is a body whose dimensions are small compared to the radius of curvature of the motion that it is undergoing like the roller coaster car or the airplane on a string described in the last lecture.

General Motion of a Particle

The motion of a particle in three-dimensional space is described by a set of coordinates that give its position as a function of time as shown in the figure below where $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors in the x, y and z directions respectively.

$$\mathbf{r} = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + z(t)\mathbf{e}_z$$



Note Unit Vector Notation

$\mathbf{i} \Leftrightarrow \mathbf{e}_x$

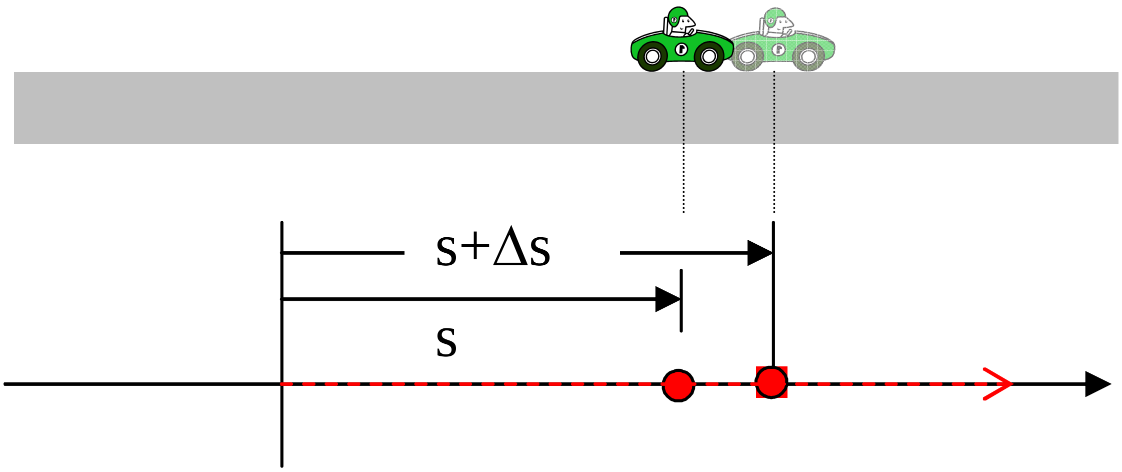
$\mathbf{j} \Leftrightarrow \mathbf{e}_y$

$\mathbf{k} \Leftrightarrow \mathbf{e}_z$

Instead of starting with this most complex case, we will first look at motions of particles on a line (rectilinear motion) and along curved paths constrained to a plane (curvilinear motion).

Rectilinear Motion of a Particle

Imagine we have a "particle" whose motion is confined to a linear path like an automobile on a straight road.



The position of the particle (or car) is given by the coordinate s(t) which is a function of time. If the particle is at position s at time t it is at a new position s+Δs at time t+ Δt. Δs can be calculated from a Taylor series expansion of s in the following way. First,

$$s(t+\Delta t) = s(t) + \frac{ds}{dt}\Delta t + \frac{1}{2}\frac{d^2 s}{dt^2}\Delta t^2 + \frac{1}{6}\frac{d^3 s}{dt^3}\Delta t^3 + ...$$

and if Δt is small we can ignore all but the first two terms (Note, if Δt =0.01, Δt^2=0.0001, etc.) and then we can estimate Δs in the following way.

$$s(t+\Delta t) - s(t) = \Delta s = \frac{ds}{dt}\Delta t$$

In the limit as Δt goes to zero this expression becomes exact. Now we know the ratio of the distance traveled to the time it took to travel that distance is just the speed of the particle so we define the speed v(t) as,

$$v(t) = \lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t} = \frac{ds}{dt}\,.$$

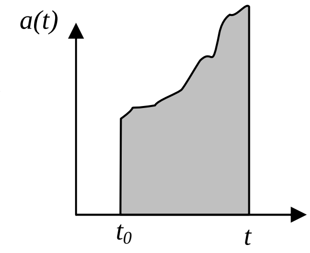
Similarly we can define the instantaneous acceleration of the particle a(t) as,

$$a(t) = \lim_{\Delta t \to 0}\frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^{s} s}{dt^2}$$

These are simple differential equations that can also be integrated (depending on what quantity is known) to determine other quantities. For example if the acceleration is known as a function of time we can obtain the other variables of the motion as follows.

$$a(t) = \frac{dv}{dt}$$

$$\therefore \int_{V_0}^{V} dv = v - v_0 = \int_{t_0}^{t} a(t)dt$$



and position can be found by integrating again,

$$v(t) = v_0 + \int_{t_0}^{t} a(t)dt$$

$$\frac{ds}{dt} = v(t)$$

$$s - s_0 = \int_{t_0}^{t} v(t)dt$$

We now take some of the simplest types of motion and solve the problem of determining one quantity from another for those cases.

Motion with Constant Acceleration

We take most simple case of constant acceleration and derive expressions for the speed and position. Constant acceleration occurs very commonly – most commonly, the motion of a body under the constant acceleration of gravity 9.8 m/s^2. This should be familiar from you early physics classes, but we repeat it here to get started.

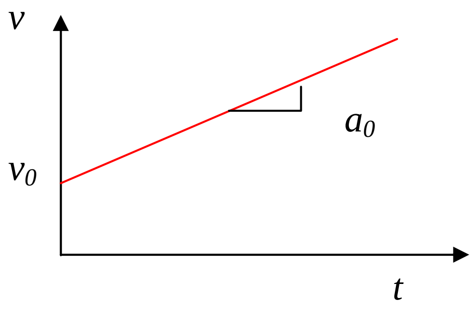
$$a(t) = \frac{dv}{dt} = a_0$$

To solve for speed we simply integrate over time once from the initial time t_0 to the time that we are interested in, t. At time t_0 the speed of the particle is known to be v_0, and at time t the speed of the particle is v. Note that without the initial information we cannot solve the problem fully – there will always be an unknown speed initial condition. Now defining the initial time to be 0, we get the following solution for speed.

$$\int_{v_0}^{v} dv = \int_{0}^{t} a_0 dt$$

$$v - v_0 = a_0 t$$

$$v = a_0 t + v_0$$

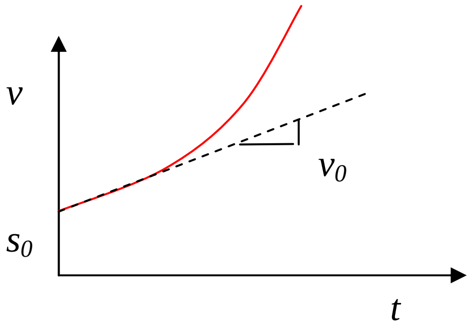


Integrating again gives the position as a function of time.

$$v = \frac{ds}{dt} = a_0 t + v_0$$

$$\int_{s_0}^{s} ds = \int_{0}^{t} (a_0 t + v_0) dt = s - s_0 = \frac{1}{2} a_0 t^2 + v_0 t$$

$$s = \frac{1}{2} a_0 t^2 + v_0 t + s_0$$



Acceleration Given as a Function of Speed

In many situations, forces in systems can be modeled as being functions of speed. This occurs in systems where we have viscous energy loss, for example in the shock absorbers of your car – pistons moving in oil. Because of Newton’s second law, this implies a speed dependent acceleration term and it is useful to be able to treat such situations mathematically. So now as a first step lets consider the case where we are given the acceleration as some arbitrary function of particle speed.

$$a = f(v)$$

How do we treat this case?

$$a = \frac{dv}{dt} = f(v)$$

$$\therefore \int_{t_0}^{t} dt = \int_{v_0}^{v} \frac{1}{f(v)} dv$$

$$t = t_0 + \int_{v_0}^{v} \frac{1}{f(v)} dv$$

$$t = t_0 + \int_{v_0}^{v} \frac{1}{f(v)} dv$$

If this can be inverted, we can solve for v(t) from this.

Alternatively, using the definitions of speed and acceleration we can also obtain the particle position as a function of speed as follows.

$$\frac{a}{v} = \frac{\frac{dv}{dt}}{\frac{ds}{dt}} = \frac{dv}{ds} = \frac{f(v)}{v}$$

$$\therefore \; ds = \frac{v}{f(v)} dv$$

$$\int_{s_0}^{s} ds = \int_{v_0}^{v} \frac{v}{f(v)} dv$$

$$s = s_0 + \int_{v_0}^{v} \frac{v}{f(v)} dv$$

$$\boxed{\frac{a}{v} = \frac{dv}{ds}}$$

Important Mathematical “Trick”

We now have the position as a function of the speed - s(v).

Acceleration Given as a Function of Position

In a third important case, the forces in a system can be described as function of displacement. This is true for elastic forces for example, also present in the shock absorber of your car – the springs. In this case we have the following.

$$a = f(s)$$

In this case, we again use the relationship from above.

$$\frac{a}{v} = \frac{dv}{ds} = \frac{f(s)}{v}$$

$$\therefore \int_{v_0}^{v} v\,dv = \int_{s_0}^{s} f(s)ds$$

$$v^2 = v_0^2 + 2\int_{s_0}^{s} f(s)ds$$

Therefore, we have the particle speed as a function of position v=g(s).
Now, we can integrate again in the following way to get position as a function of time.

$$v = \sqrt{v_0^2 + 2\int_{s_0}^{s} f(s)ds}$$

$$\frac{ds}{dt} = \sqrt{v_0^2 + 2\int_{s_0}^{s} f(s)ds}$$

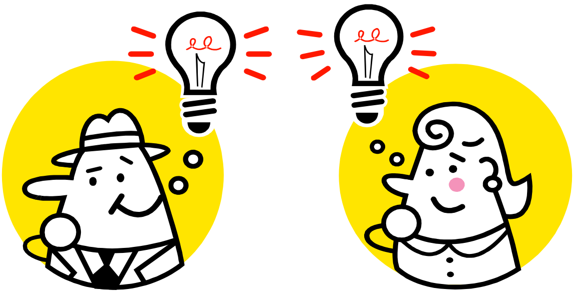
$$\int_{t_0}^{t} dt = \int_{s_0}^{s} \frac{1}{\sqrt{v_0^2 + 2\int_{s_0}^{s} f(s)ds}}\,ds$$

$$t - t_0 = \int_{s_0}^{s} \frac{1}{\sqrt{v_0^2 + 2\int_{s_0}^{s} f(s)ds}}\,ds$$

If this mess can be inverted, we can solve for position as a function of time s(t).

In the next lecture, we will do several example problems to illustrate the theory described here.

Important: Stop and Think!



Ask Yourself the Following Concept Questions…

Q1: In one case, you are sitting in your car, but you have not pushed the accelerator and you are not moving. In a second case, you are sitting in your car and you push the accelerator petal as far as it will go, but you are not not yet moving. You know this because the speedometer still reads zero right when you push the accelerator. In both cases you are not moving. But … is there a difference between the two cases? (Hint: What is the name of the petal you pushed in case two but not in case one?)

Q2: Can an object have a speed of zero but an acceleration that is not zero?

Q3: A car has a apeed that is increasign linearly with time so that it can be described by the following mathematical equation.

$$v = 19.6t$$

What is the speed of the object at time t=0?

What is the accleration of the object at time t=0? About how many g's of accleration is this? What would it feel like to be riding on the object at time t=0?

Bring it together: Are your answers to the three questions above consistent with each other? Try to connect the reality of being in your car with what the mathematics is telling you. If you don't believe it, try the experiement ina parking lot…just don't get a ticket for "peeling out". A related question to think about is what is the speed of a superball at the instant it hits the floor? What is its acceleration? (Hint: If there is an unblanced force on something then it is accelerating.)

Flashback to Physics Class

Q1. What is the difference between "speed" and "velocity"?